J. Fluid Mech. (1985), *vol.* 155, *pp.* 327–341
Printed in Great Britain

Conically similar viscous flows. Part 1. Basic conservation principles and characterization of axial causes in swirl-free flow

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(Received 2 February 1984 and in revised form 31 December 1984)

This paper has three main objectives. First, it aims to show that basic general conservation principles for viscous flow can be formulated in terms of diffusion and convection. Secondly, it aims to show that three scalar conservation principles suffice to provide a method for characterizing swirling axisymmetric flows in terms of axial and boundary production of the conserved quantities. Thirdly, it aims to exemplify these two objectives by giving a complete specification of the axial causes for swirl-free conically similar flow in otherwise free space.

This series of papers, overall, is concerned with the analysis and characterization of swirling conically similar flows in terms of the singularities that generate the conserved quantities. In conically similar flows there is no natural lengthscale, and the sole parameters governing the flow are provided by the strengths of the singularities that cause the flow. These are required to have the same dimensions as a power of the kinematic viscosity ν. The axisymmetric flow generated by uniform production of swirl angular momentum per unit mass along a half-axis at a constant rate provides a simple example.

In conically similar flow the three conservation principles for axisymmetric flow provide a sixth-order non-autonomous system of two ordinary differential equations governing the flow. Here, in Part 1, these equations are derived for the general case of swirling flow, and are shown to reduce to a fourth-order system when swirl is absent. The two scalar conservation principles describing swirl-free flow are used to classify the basic axial causes for this system.

Part 2 analyses these basic exact one-parameter swirl-free families of solutions, and Part 3 extends the analysis to the remaining one-parameter family of swirling flows associated with uniform swirl angular-momentum production on a half-axis. Each of the families is characterized by a single independent cause, and two of them provide new non-trivial solutions of the Navier–Stokes equations. The effects of nonlinear coupling of these basic one-parameter causes and of conically similar distributions over conical boundaries will be examined in later papers.

1. Introduction

The motion of an incompressible homogeneous viscous fluid under conservative body forces can be simply described in terms of two basic conservation principles – conservation of volume and conservation of whirl. The amount of *whirl* in a fixed region V at time t is defined to be the vector quantity $\boldsymbol{\Gamma}$ given by

$$\boldsymbol{\Gamma}(V,t) = \int_V \boldsymbol{\omega}\,\mathrm{d}V. \tag{1.1}$$

Here $\boldsymbol{\omega}(\boldsymbol{r}, t)$, its volume density at time t, is the vorticity field, which is related to the velocity field $\boldsymbol{q}(\boldsymbol{r}, t)$ by

$$\operatorname{curl} \boldsymbol{q} = \boldsymbol{\omega}. \tag{1.2}$$

Locally, conservation of volume requires

$$\operatorname{div} \boldsymbol{q} = 0, \tag{1.3}$$

whilst Helmholtz's equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \operatorname{div}\left[\boldsymbol{q}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{q} - \nu \nabla \boldsymbol{\omega}\right] = 0 \tag{1.4}$$

specifies conservation of whirl by relating the time rate of change of its vector volume density to a tensor flux which contains both convective and diffusive terms, written in a coordinate-free dyadic form. Here ν is the (constant) kinematic viscosity.

Whirl is generated in the fluid by the action of non-conservative forces. Usually it is produced by the action of skin friction at rigid boundaries, but wider circumstances can be contemplated in which it is produced at singular points, lines and surfaces by arbitrary external causes. Conservation equations can also be formulated for both the first and second moment of whirl about a chosen origin O, and these provide a means of characterizing higher-order whirl-producing singularities. In this approach, which we describe as the *kinematic* approach, the dynamical role of the pressure field $p(\boldsymbol{r}, t)$ is relegated to the equations of motion, which provide its gradient field in terms of $\boldsymbol{q}$ and $\boldsymbol{\omega}$ and the constant density ρ.

The induced (Biot–Savart) velocity field, generated by the vorticity field $\boldsymbol{\omega}$, is determined by the appropriate particular solution of (1.2). To this must be added the potential-flow field generated by the volume-producing singularities. Outside these singularities this flow field satisfies both (1.3) and the homogeneous form of (1.2) with $\boldsymbol{\omega}$ zero. The problem is nonlinear in that the convective terms contained in the whirl flux tensor depend, in part, upon the induced velocity field caused by the whirl distribution.

In axisymmetric flow the above basic vector conservation principles simplify to provide a set of three governing scalar conservation principles for volume, ring circulation and swirl angular momentum (Pillow 1970). The volume densities of these latter two quantities are denoted by $l/2\pi$ and ρT respectively, and are simply related to azimuthal components of the vorticity and velocity fields.

In cylindrical polar coordinates (x, σ, ϕ) with x measured along the axis of symmetry, σ perpendicular to it and ϕ the azimuthal angle (so that the tangent vectors to the coordinate lines $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}})$ form a right-handed orthonormal triad with natural basis vectors $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\sigma}}, \sigma\hat{\boldsymbol{\phi}})$), the velocity $\boldsymbol{q}$ and the vorticity $\boldsymbol{\omega}$ can be written in the form

$$\boldsymbol{q} = \boldsymbol{u} + \frac{T}{\sigma}\hat{\boldsymbol{\phi}}, \tag{1.5}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \sigma l \hat{\boldsymbol{\phi}}. \tag{1.6}$$

Here the axial half-plane components of the velocity and vorticity are $\boldsymbol{u}$ and $\boldsymbol{\Omega}$ respectively, and $2\pi T$ is the swirl circulation. Also,

$$\operatorname{curl} \boldsymbol{u} = \sigma l \hat{\boldsymbol{\phi}}, \tag{1.7}$$

whilst

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.8}$$

and

$$\operatorname{div} \boldsymbol{\Omega} = 0. \tag{1.9}$$

For a ring R, formed by rotating its axial half-plane cross-section C about the axis of symmetry, Stokes' theorem requires that

$$\int_R \frac{l}{2\pi}\,\mathrm{d}V = \int_C \hat{\boldsymbol{\phi}}\cdot\operatorname{curl}\boldsymbol{u}\,\mathrm{d}S = \oint_{\partial C}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{r}. \tag{1.10}$$

The last line integral describes the circulation in an axial half-plane around the boundary ∂C of the cross-section. This circulation, which we define as *ring circulation*, is independent of ϕ, and (1.10) reveals that it may be regarded as being uniformly distributed with respect to ϕ around the ring. The volume density of ring circulation is $l/2\pi$. Equation (1.8) shows that the axial half-plane velocity $\boldsymbol{u}$ may be described in terms of a scalar flux function, which in this case is the stream function ψ. The vector potential for $\boldsymbol{u}$ is thus $(\psi/\sigma)\hat{\boldsymbol{\phi}}$:

$$\boldsymbol{u} = \operatorname{curl}\left(\frac{\psi}{\sigma}\hat{\boldsymbol{\phi}}\right) = \frac{1}{\sigma}\psi_\sigma\hat{\boldsymbol{x}} - \frac{1}{\sigma}\psi_x\hat{\boldsymbol{\sigma}}. \tag{1.11}$$

The increment in $2\pi\psi$ along any axial half-plane arc then measures the volume discharge per unit time from left to right across the surface generated by the rotation of that arc about the axis of symmetry. Similarly, in accord with (1.9), the flux function for the axial half-plane vorticity $\boldsymbol{\Omega}$ is T, and

$$\boldsymbol{\Omega} = \operatorname{curl}\left(\frac{T}{\sigma}\hat{\boldsymbol{\phi}}\right) = \frac{1}{\sigma}T_\sigma\hat{\boldsymbol{x}} - \frac{1}{\sigma}T_x\hat{\boldsymbol{\sigma}}. \tag{1.12}$$

The axial, half-plane, flow field $\boldsymbol{u}$ is completely determined by the stream function ψ. The relationship between ψ and l is obtained by combining (1.7) with (1.11) to yield

$$\operatorname{curl}\operatorname{curl}\left(\frac{\psi}{\sigma}\hat{\boldsymbol{\phi}}\right) = \sigma l\hat{\boldsymbol{\phi}}, \tag{1.13}$$

or, in scalar form,

$$\psi_{xx} - \frac{1}{\sigma}\psi_\sigma + \psi_{\sigma\sigma} = -\sigma^2 l. \tag{1.14}$$

A knowledge of the distribution of volume-producing singularities allows the velocity field $\boldsymbol{q}$ and the stream function ψ to be reconstructed from the scalar fields l and T by adding the potential flow of the volume production to the ring-circulation-induced velocity field determined by (1.7) and (1.8).

The conservation equations for ring circulation and swirl angular momentum in axisymmetric flow are formulated in §2, and a detailed interpretation of the individual terms in their flux vectors is given there. Under steady-state conditions, both these flux vectors are solenoidal and scalar flux functions can again be introduced to measure the discharge per unit time of these conserved quantities across axisymmetric surfaces.

Conically similar viscous flows are a simple, special class of axisymmetric flows in which the sole parameters characterizing the flow causes are of the same dimensions as powers of ν. Since no lengthscale is then available, the radial dependence of physical fields can be directly determined by dimensional analysis in spherical polar coordinates (r, μ, ϕ). Here (r, θ) are polar coordinates in an axial half-plane, $\mu = \cos\theta$

and $\theta = 0$ on the positive x-axis. In particular, for steady-state flow, the field quantities ψ, l and T can be written in the form

$$\psi = \nu r f(\mu), \tag{1.15}$$

$$l = \frac{\nu}{r^3} g(\mu), \tag{1.16}$$

$$T = \nu \tau(\mu), \tag{1.17}$$

where f, g and τ are dimensionless.

For such ψ and l, (1.14), when rewritten in polar coordinates, requires that

$$f'' = -g. \tag{1.18}$$

Because of the known radial dependences, the axisymmetric scalar conservation principles now reduce to a set of two coupled nonlinear ordinary differential equations for f and τ, as shown in §3. The system of governing equations is a sixth-order, non-autonomous one. It still features all the physically different terms which contribute to the flux vectors of the conserved quantities. Conically similar viscous flows thus effectively illustrate the axisymmetric conservation principles, and are used throughout this set of papers to demonstrate their central importance.

The viewpoint of the present series of papers is that, since axisymmetric flows are described fully by three scalar conservation principles, it should be possible to specify conically similar viscous flows in terms of the conically similar production of these three basic conserved quantities on conical boundaries or on the axis of the flow. Since most flows with conical boundaries may be continued to the axis of symmetry, the simplest approach is to specify first only the *axial* causes. This present paper describes such a characterization in §4 for the case of swirl-free flows. Part 2 examines the flows corresponding to the individual one-parameter swirl-free axial causes. Part 3 generalizes the characterization to flows with swirl, and details the remaining fundamental flow whose cause is a uniform distribution of swirl angular-momentum sources on a half-axis.

Useful reviews of known swirl-free conically similar viscous flows have been given, for example, by Whitham (1963), Berker (1963) and Yatseyev (1950). However, although these flows are relevant to our later studies of nonlinear coupling, as yet, no systematic characterization has become available.

In swirling conically similar flow the three basic scalar conservation principles specify a total of six independent axial causes, combinations of which are sufficient to characterize conically similar viscous flows without boundaries. Two new flows, one with swirl, the other swirl-free, are uncovered by this approach. In the swirling example (§4 of Part 3) large swirl angular-momentum production on a half-axis leads to a well-developed internal boundary layer in the form of a radial jet in a thin conical layer separating swirl-free and constant-swirl-circulation regions. In the new swirl-free flow (§4 of Part 2) physical production of a second moment of ring circulation along the axis of symmetry produces opposed axial jets. When directed inwards, these erupt and discharge into a plane radial jet. Outwardly directed jets result in induced outer potential flow towards the axis of symmetry.

The flow produced by a point source of axial momentum (equivalent to production of a second moment of ring circulation) (Landau 1943; Squire 1951) and the flow associated with a uniform half-line source of mass naturally arise afresh in the classification proposed and complete the basic set of fundamental flows.

2. Kinematic conservation principles for axisymmetric viscous flow

Three basic kinematic conservation principles suffice to describe axisymmetric incompressible viscous flow (Pillow 1970). They are presented here in somewhat modified form, but still concern the conservation of volume, ring (axial half-plane) circulation, and kinematic swirl angular momentum (with volume density T).

Vector conservation principles with antisymmetric flux tensors are an abundant source of scalar conservation principles, since contravariant components in arbitrary curvilinear coordinate systems are conserved (Paull 1982). The flux tensor for whirl in (1.4) can be rendered antisymmetric by the addition of a zero-divergence flux tensor $\nu(\nabla\boldsymbol{\omega})^{\dagger}$;

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\operatorname{div}\boldsymbol{W}=0, \quad \text{where} \quad \boldsymbol{W}=\boldsymbol{q}\boldsymbol{\omega}-\boldsymbol{\omega}\boldsymbol{q}-\nu\nabla\boldsymbol{\omega}+\nu(\nabla\boldsymbol{\omega})^{\dagger}. \tag{2.1}$$

(Here $(\nabla\boldsymbol{\omega})^{\dagger}$ denotes the transpose of $\nabla\boldsymbol{\omega}$.) This is the form of the conservation principle for whirl adopted in this series of papers.

One scalar conservation principle derived from (2.1) results from taking the natural azimuthal contravariant component. This gives the conserved volume density for ring circulation:

$$l=\boldsymbol{\omega}\cdot\nabla\phi=\boldsymbol{\omega}\cdot\left(\frac{1}{\sigma}\hat{\boldsymbol{\phi}}\right)=(\nabla\times\boldsymbol{u})\cdot\left(\frac{1}{\sigma}\hat{\boldsymbol{\phi}}\right). \tag{2.2}$$

As a consequence of the antisymmetric flux tensor for whirl in (2.1), conservation of ring circulation is thus governed by the flux vector $\boldsymbol{W}\cdot\nabla\phi$. The flux vector for ring circulation is then $\boldsymbol{J}/2\pi$, and is related to the ring-circulation volume density $l/2\pi$ by

$$\frac{\partial l}{\partial t}+\operatorname{div}\boldsymbol{J}=-4\pi\nu l\delta(\boldsymbol{\sigma}), \tag{2.3}$$

where

$$\boldsymbol{J}=l\boldsymbol{q}-\frac{T}{\sigma}\frac{\boldsymbol{\omega}}{\sigma}-2l\boldsymbol{q}_0-\nu\nabla l, \tag{2.4}$$

and

$$\boldsymbol{q}_0=\frac{\nu\hat{\boldsymbol{\sigma}}}{\sigma}. \tag{2.5}$$

Here $\boldsymbol{q}_0$ is the velocity field of a viscosity-dependent line source on the axis of symmetry with uniform line density $2\pi\nu$, and $\delta(\boldsymbol{\sigma})$ is the two-dimensional delta function concentrated on the axis of symmetry, with $\boldsymbol{\sigma}=\sigma\hat{\boldsymbol{\sigma}}$. The singularity on the right-hand side of (2.3) appears in addition to the physical causes of the flow, and results from

$$\operatorname{div}\boldsymbol{q}_0=2\pi\nu\delta(\boldsymbol{\sigma}). \tag{2.6}$$

In swirl-free flow, where $T=0$, $\boldsymbol{q}=\boldsymbol{u}$, $\boldsymbol{\Omega}=0$ and $\boldsymbol{\omega}=l\sigma\hat{\boldsymbol{\phi}}$, only three processes contribute to the flux vector $\boldsymbol{J}/2\pi$, and its description is pleasingly simple:

$$\frac{1}{2\pi}\boldsymbol{J}=\frac{1}{2\pi}[l\boldsymbol{u}-2l\boldsymbol{q}_0-\nu\nabla l]. \tag{2.7}$$

The first term $l\boldsymbol{u}$ in $\boldsymbol{J}$ represents the convective flux of positive (anticlockwise when viewed *from* the $\hat{\phi}$-direction) ring circulation by the actual axial half-plane velocity field $\boldsymbol{u}$. The last term $-\nu\nabla l$ in $\boldsymbol{J}$ describes the diffusive flux of ring circulation down the gradient of its density. The second term $-2l\boldsymbol{q}_0$ describes a uniform convective suction of ring circulation towards the axis of symmetry arising from an *apparent* uniform line sink of volume there of strength $4\pi\nu$ per unit length. This viscosity-dependent fictitious convection field is *not* present in the actual field of flow, but (2.7) shows that ring circulation is indeed convected as if it were. This process (which may be conveniently described as *viscous convection*) sweeps ring circulation towards the axis of symmetry, where it is destroyed by the singular terms on the right-hand side of (2.3). The term $-4\pi\nu l\delta(\boldsymbol{\sigma})$ describes a line sink of ring circulation of variable strength along the axis, which, in the absence of physical causes and boundaries, is just sufficient to remove the ring circulation brought there by viscous convection.

In swirling flow a further process associated with distributed production of ring circulation by the swirl velocity gradient comes into play. There is rotation and extension of the vortex tubes, and the flux vector may be put into the form

$$\frac{1}{2\pi}\boldsymbol{J} = \frac{1}{2\pi}\left[l\boldsymbol{q} - \frac{T}{\sigma}\frac{\boldsymbol{\omega}}{\sigma} - 2l\boldsymbol{q}_0 - \nu\nabla l\right]. \tag{2.8}$$

The new term $-T\boldsymbol{\omega}/\sigma^2$ in $\boldsymbol{J}$ describes the effect of transporting whirl by the swirl velocity. In this interpretation, vortex filaments are split into their axial half-plane and azimuthal components ($\boldsymbol{\omega} = \boldsymbol{\Omega} + \sigma l\hat{\boldsymbol{\phi}}$). Azimuthal transport of whirl, as described by $-T\boldsymbol{\Omega}/\sigma^2$, leads in general to production of ring circulation as a result of rotation of the axial half-plane vorticity out of that plane, whilst the azimuthal transport of ring circulation, which results in possible extension of ring vortices, is accounted for by $-Tl\hat{\boldsymbol{\phi}}/\sigma$. Collectively, the two processes give rise to a flux $-T\boldsymbol{\omega}/\sigma^2$ for clockwise ring circulation down the vortex tubes. This of course follows directly from the term $-\boldsymbol{\omega}\boldsymbol{q}$ displayed in the whirl flux tensor of (2.1), where it describes the flux of vector whirl up the vortex tubes (Paull 1982), just as the flux of whirl down the stream tubes is described by $\boldsymbol{q}\boldsymbol{\omega}$. (The dyad $-\boldsymbol{\omega}\boldsymbol{q}$ provides a flux description of the localized rate of production of whirl (with volume density $\boldsymbol{\omega}\cdot\nabla\boldsymbol{q}$) associated with the rotation and stretching of vortex tubes which would otherwise appear on the right-hand side of (2.1).)

It should also be noted that the convective term $l\boldsymbol{q}$ in $\boldsymbol{J}$ now contains an azimuthal component which is exactly cancelled by the azimuthal component of the vortex-tube flux. It therefore follows that $\boldsymbol{J}$ always lies entirely in the axial half-plane, and may be written in the form

$$\boldsymbol{J} = l\boldsymbol{u} - \frac{T}{\sigma}\frac{\boldsymbol{\Omega}}{\sigma} - 2l\boldsymbol{q}_0 - \nu\nabla l, \tag{2.9}$$

which extends the swirl-free description given in (2.7). The new term $-T\boldsymbol{\Omega}/\sigma^2$ describes a flux of clockwise ring circulation in the axial half-plane along the lines of constant swirl circulation (T), which are the flux lines of the axial half-plane vorticity field $\boldsymbol{\Omega}$.

The above kinematic interpretation of the flux vector in the ring-circulation conservation equation (2.3) provides a useful and fully consistent alternative to the one suggested by Pillow (1970), where $(-T^2/\sigma^4)\hat{\boldsymbol{x}}$ replaces the term $-T\boldsymbol{\Omega}/\sigma^2$ in $\boldsymbol{J}$ above. The new interpretation has the advantage that there is no flux of ring circulation in constant-swirl-circulation flows, though both interpretations need to be modified by subtraction of an axial flux in solid-body rotation.

The axial half-plane component of the conservation-of-whirl equation (2.1) in the vector form

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} + \nabla\cdot[\boldsymbol{q}\boldsymbol{\Omega} - \boldsymbol{\omega}\boldsymbol{u} - \nu\{\nabla\boldsymbol{\Omega} - (\nabla\boldsymbol{\Omega})^{\dagger}\}] = 0 \tag{2.10}$$

yields a further independent scalar conservation principle when swirl is present. Since the swirl circulation $2\pi T$ provides a flux function for the axial half-plane vorticity field $\boldsymbol{\Omega}$, changes in T are related to displacements $\mathrm{d}\boldsymbol{r}$ along arcs in the axial half-plane by

$$\mathrm{d}T = \boldsymbol{\Omega}\cdot \mathrm{d}\boldsymbol{r}\times \sigma\hat{\boldsymbol{\phi}}. \tag{2.11}$$

In the absence of swirl circulation on the x-axis, integration of (2.10) along any half-plane path starting from the origin (say) provides, after some manipulation, the governing conservation equation for a quantity whose volume density is T;

$$\frac{\partial T}{\partial t}+\operatorname{div}\boldsymbol{K} = 0, \tag{2.12}$$

where

$$\boldsymbol{K} = T\boldsymbol{u}+2T\boldsymbol{q}_0-\nu\nabla T. \tag{2.13}$$

Equation (2.12), as noted by Pillow (1970), follows directly from the covariant swirl component of the momentum equation, since the azimuthal component of the gradient of the Bernoulli function B will be zero in axisymmetric flow if B is single-valued. (A multivalued B would describe swirl circulation production on the axis.) Here

$$B = \frac{p}{\rho}+\tfrac{1}{2}q^2+\Phi, \tag{2.14}$$

p is the pressure excess over that at infinity and Φ is the conservative body force potential per unit mass.

In the scalar conservation principle (2.12), T plays the role of a volume density. Since ρT is the swirl angular-momentum density, T itself will be called the *kinematic swirl angular-momentum* volume density. Again there are just three processes that contribute to the flux vector $\boldsymbol{K}$ for kinematic swirl angular momentum. The first term $T\boldsymbol{u}$ in $\boldsymbol{K}$ represents a convective flux along the constant-ψ lines in an axial half-plane, whilst the third term $-\nu\nabla T$ describes the diffusive flux of kinematic swirl angular momentum down its density gradient. The second term $2T\boldsymbol{q}_0$ is again a viscous fictitious convective flux, but now it is outwardly directed from the axis and arises from an apparent uniform line source of volume there of strength $4\pi\nu$ per unit length. In the absence of external causes, T is zero on the axis, and there is consequently no need for a singular source term on the right-hand side of (2.12) analogous to the sink term that appears in (2.3). Such a term $4\pi\nu T\delta(\sigma)$ would be necessary, however, if angular momentum were supplied by couples distributed along the axis with constant line density ρT. Indeed, the constant-T solution provides a simple example of how, in a viscous fluid, angular momentum is transmitted to infinity by viscous convection from a maintained potential-flow line vortex on the x-axis arising from the application of a uniform couple along this axis.

If the steady Navier–Stokes equation in the form

$$\boldsymbol{\omega}\times\boldsymbol{q}+\nu(\nabla\times\boldsymbol{\omega})+\nabla B = 0 \tag{2.15}$$

is cross-multiplied with the identity tensor, it follows that the flux tensor $\boldsymbol{W}$ for whirl is given by

$$\boldsymbol{W} = \boldsymbol{q}\boldsymbol{\omega}-\boldsymbol{\omega}\boldsymbol{q}-\nu[\nabla\boldsymbol{\omega}-(\nabla\boldsymbol{\omega})^{\dagger}] = -\boldsymbol{I}\times\nabla B. \tag{2.16}$$

Since $l = \boldsymbol{\omega}\cdot\nabla\phi$, the flux vector $\boldsymbol{J}/2\pi$ for ring circulation in steady flow must be given by

$$\boldsymbol{J} = -\boldsymbol{I}\times\nabla B\cdot\frac{\hat{\boldsymbol{\phi}}}{\sigma} = \nabla\times\left[-B\frac{\hat{\boldsymbol{\phi}}}{\sigma}\right]. \tag{2.17}$$

In steady flow, away from the axis of symmetry, $\boldsymbol{J}$ is solenoidal and allows the introduction of a vector potential. Equation (2.17) shows that $-B\hat{\boldsymbol{\phi}}/\sigma$ is a suitable vector potential and that the negative Bernoulli function $-B/2\pi$ plays the role of a flux function for ring circulation. Apart from its dynamical role, B also has this kinematic role in that its decrease along any axial half-plane arc measures the rate of discharge of ring circulation (from left to right) across the surface generated by rotation of the arc about the axis of symmetry.

Similarly, since $\boldsymbol{K}$ is solenoidal in steady flow, it has a vector potential $\Lambda\hat{\boldsymbol{\phi}}/\sigma$, and

$$\boldsymbol{K} = T\boldsymbol{u}+2T\boldsymbol{q}_0-\nu\nabla T = \nabla\times\left(\frac{\Lambda}{\sigma}\hat{\boldsymbol{\phi}}\right). \tag{2.18}$$

Again, the new function Λ plays the role of a flux function for kinematic swirl angular momentum.

Each of the three conserved quantities, volume, ring circulation and kinematic swirl angular momentum, in axisymmetric flow can be generated independently of the others. Volume-producing singularities do not produce ring circulation or kinematic swirl angular momentum. This is one of the advantages of the kinematical approach over the dynamical one. In the conventional approach, conservation of axial momentum is used instead of conservation of ring circulation. Forces (i.e. sources of axial momentum) are required to hold volume-producing singularities in place, and inevitably the strengths of the two causes are then linked.

3. Conically similar viscous flow

The ordinary differential equations governing conically similar viscous flow can be simply derived from the basic axisymmetric ones (1.13), (2.3) and (2.12) when they are cast in spherical polar form. Thus, away from the axis of symmetry, in steady flow

$$l = -\frac{1}{r^4(1-\mu^2)}[(1-\mu^2)\,\psi_{\mu\mu}+r^2\psi_{rr}], \tag{3.1}$$

$$\frac{\partial(\psi,l)}{\partial(r,\mu)}-\nu[(1-\mu^2)\,l_{\mu\mu}-4\mu l_\mu+r^2l_{rr}+4rl_r]-\frac{2TT_\mu}{r^3(1-\mu^2)}-\frac{2\mu TT_r}{r^2(1-\mu^2)^2}=0 \tag{3.2}$$

and

$$\frac{\partial(\psi,T)}{\partial(r,\mu)}-\nu[(1-\mu^2)\,T_{\mu\mu}+r^2T_{rr}]=0. \tag{3.3}$$

Here $r^2 = x^2+\sigma^2$ and $\mu = x/r$.

If the ψ-, l- and T-fields are replaced in (3.1), (3.2) and (3.3) by their conically similar expressions $\nu rf(\mu)$, $\nu g(\mu)/r^3$ and $\nu\tau(\mu)$ respectively in accord with (1.15), (1.16) and (1.17), then the following ordinary differential equations result:

$$g = -f'', \tag{3.4}$$

$$fg'+3f'g-[(1-\mu^2)\,g''-4\mu g']-\frac{2\tau\tau'}{1-\mu^2}=0, \tag{3.5}$$

$$f\tau'-(1-\mu^2)\,\tau''=0. \tag{3.6}$$

Primes represent differentiation with respect to μ. The relation (3.4) allows (3.5) to be integrated three times. The governing ordinary integrodifferential system is then

$$(1-\mu^2)f'+2\mu f-\tfrac{1}{2}f^2 = G(\mu,\tau^2)+A\mu^2+B\mu+C \tag{3.7}$$

and
$$(1-\mu^2)\tau''-f\tau' = 0, \tag{3.8}$$

where
$$G(\mu,\tau^2) = \int^{\mu} \mathrm{d}\xi \int^{\xi} \mathrm{d}\eta \int^{\eta} \frac{2\tau(\alpha)\,\tau'(\alpha)}{1-\alpha^2}\mathrm{d}\alpha \tag{3.9}$$

and A, B and C are constants of integration. Analogous equations have previously been derived by Gol'dshtik (1960) and Serrin (1972) from the dynamical equations of motion. If their lead is followed, the expression for $G(\mu,\tau^2)$ may be integrated by parts three times and rewritten as

$$G(\mu,\tau^2) = -\left(\frac{1-\mu}{2}\right)^2 \int_{\mu_1}^{\mu} \frac{\tau^2(\xi)\,\mathrm{d}\xi}{(1-\xi)^2} - \left(\frac{1+\mu}{2}\right)^2 \int_{\mu}^{\mu_0} \frac{\tau^2(\xi)\,\mathrm{d}\xi}{(1+\xi)^2}, \tag{3.10}$$

where $\mu \in [\mu_1, \mu_0]$ defines the region of flow. Equation (3.7) is of Riccati type and hence can also be partially linearized by the substitution

$$f = -2(1-\mu^2)\frac{h'}{h}. \tag{3.11}$$

The system of governing equations then becomes

$$h''(\mu) + \frac{[G(\mu,\tau^2)+A\mu^2+B\mu+C]}{2(1-\mu^2)^2}h(\mu) = 0 \tag{3.12}$$

and

$$\tau(\mu) = D\int_0^{\mu} \frac{\mathrm{d}\xi}{h^2(\xi)} + E, \tag{3.13}$$

where D and E are further constants of integration. The function $h(\mu)$ is specified to within a constant multiple. It specifies a potential function ϕ_{T} for the transverse $\hat{\boldsymbol{\theta}}$-velocity, where

$$\phi_{\mathrm{T}} = -2\nu \ln h. \tag{3.14}$$

The fractional gain in $h(\mu)$ along a meridional arc on any sphere centred on the origin describes an increase in ϕ_{T} which specifies the line integral of $\boldsymbol{q}$ along that arc (the transverse circulation).

The flux function ψ for volume is $\nu r f(\mu)$, and its flux vector the velocity $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = -\frac{\nu}{r}\left[f'(\mu)\,\hat{\boldsymbol{r}} + \frac{f(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\hat{\boldsymbol{\theta}} - \frac{\tau(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\hat{\boldsymbol{\phi}}\right]. \tag{3.15}$$

For ring circulation, the flux function is $-B/2\pi$, where B is the Bernoulli function. For conically similar viscous flows, B must have the form

$$B = \frac{\nu^2}{r^2}\beta(\mu). \tag{3.16}$$

The steady Navier–Stokes equation (2.15) then yields

$$\beta'(\mu) = (1-f')g + \frac{\tau\tau'}{1-\mu^2} \tag{3.17}$$

and

$$-2\beta(\mu) = [(1-\mu^2)g]' - fg. \tag{3.18}$$

Equation (3.7) can be differentiated twice to give the alternative expression

$$-2\beta(\mu) = (2-f')f' - 2A - G''(\mu,\tau^2). \tag{3.19}$$

The flux vector for ring circulation is

$$\frac{1}{2\pi}\boldsymbol{J} = \frac{\nu^2}{2\pi r^4}\left\{\beta'(\mu)\,\hat{\boldsymbol{r}} - \frac{2\beta(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\hat{\boldsymbol{\theta}}\right\}. \tag{3.20}$$

For kinematic swirl angular momentum, the flux function Λ has the form

$$\Lambda = \nu^2 r \lambda(\mu). \tag{3.21}$$

Equation (2.18) then gives

$$-\lambda'(\mu) = (2-f')\,\tau \tag{3.22}$$

and

$$-\lambda(\mu) = (1-\mu^2)\,\tau' + 2\mu\tau - f\tau. \tag{3.23}$$

The flux vector for kinematic swirl angular momentum is given by

$$\boldsymbol{K} = \frac{\nu^2}{r}\left[-\lambda'(\mu)\,\hat{\boldsymbol{r}} - \frac{\lambda(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\hat{\boldsymbol{\theta}}\right]. \tag{3.24}$$

4. Quantitative characterization of axial causes in swirl-free conically similar flow

The basic conservation principles for volume and ring circulation developed in §2 are used in this section to determine the individual strengths of the swirl-free axial causes of conically similar viscous flow. The ability to describe the strengths of the basic singularities allows any particular combination of causes to be specified and investigated at will, and so provides a means of classifying and labelling swirl-free conically similar viscous flows in terms of the strengths of their constituent causes. Completeness of the fundamental set of swirl-free *axial* causes, within the class of solutions described by (1.15) and (1.16), is guaranteed by the determination of the three constants of integration in the governing Riccati equation (3.7) and the imposition of an integral condition on the flow.

Creation of ring circulation on the axis calls for singular production of ring circulation. This is measured by the rate of production of the axial component of moment of whirl. It will be shown that the complete set of swirl-free causes on the axis is provided by two independent uniform half-line volume sources, one point source of the axial component of moment of whirl (axial momentum) and one antisymmetric distribution about the origin of sources of the axial component of moment of whirl (axial momentum) with line density inversely proportional to x. The flows generated by each of the basic causes, listed above, are examined in Part 2.

In swirl-free conically similar viscous flows the governing equations (3.7)–(3.9) reduce to

$$(1-\mu^2)f' + 2\mu f - \tfrac{1}{2}f^2 = A\mu^2 + B\mu + C, \tag{4.1}$$

since the swirl circulation T is everywhere-zero.

If volume sources are distributed uniformly along the half-axes $\mu = \pm 1$ with line density strength $M_{\pm 1}$ then conically similar flows are generated, since the constants $M_{\pm 1}$ describe the volume emitted per unit time per unit length on each half-axis and have the dimensions of ν. For such distributions, ψ varies linearly with r on each half-axis, and (1.15) requires

$$M_{\pm 1} = \mp 2\pi\nu f(\pm 1). \tag{4.2}$$

Analytic solutions of (4.1) for which f approaches a constant as $\mu \to \pm 1$ must also fulfil the requirement $(1-\mu^2)f' \to 0$ as $\mu \to \pm 1$. Hence, for finite $M_{\pm 1}$,

$$\pm 2f(\pm 1) - \tfrac{1}{2}f^2(\pm 1) = A \pm B + C. \tag{4.3}$$

Thus

$$A + C = [f(1) - f(-1)] - \tfrac{1}{4}[f^2(1) + f^2(-1)] \tag{4.4}$$

and

$$B = [f(1) + f(-1)]\,[1 - \tfrac{1}{4}\{f(1) - f(-1)\}], \tag{4.5}$$

where

$$f(\pm 1) = \mp M_{\pm 1}/2\pi\nu.$$

Consideration of the conically similar flow generated by ring-circulation production on the cone $\mu = \mu_0$ with area density inversely proportional to $r\sigma^3$ makes it clear that, in the limiting case as $\mu_0 \to 1$, the line density of ring-circulation production on $\mu = 1$ must be infinite if non-trivial flows are to result. However, the line density of the second (σ^2) moment of ring circulation production is a finite quantity in this limit and will produce a conically similar flow if it varies inversely with x, since the strength of this line singularity then has the dimensions of ν^2. An even simpler conically similar flow results if the second (σ^2) moment of ring circulation is produced at the origin at a finite rate L per unit time. Such a cause may be regarded as the limit of axisymmetric ring-circulation production at the rate Q per unit time on the ring through $(0, \sigma_0)$ as $\sigma_0 \to 0$, with $L = Q\sigma_0^2$ held finite. Again L has the dimensions of ν^2 and provides a simple kinematic characterization of the point source of axial momentum and the conically similar flow associated with it by Landau (1943) and Squire (1951).

This series of papers views conservation of ring circulation as the basic and simplest kinematic principle describing the axial half-plane distribution of the $\hat{\boldsymbol{\phi}}$-component of the vorticity field. However, for the purpose of *measuring* the strengths of the axial ring-circulation singularities, it is evident that a σ^2 moment of the conservation principle for ring circulation needs to be developed. Conservation of the first moment of whirl provides such a principle.

If the vector moment $\boldsymbol{M}_0$ of whirl about the origin O in a fixed region V is defined as

$$\boldsymbol{M}_0(V, t) = \int_V (\boldsymbol{r} \times \boldsymbol{\omega})\, \mathrm{d}V, \tag{4.6}$$

then a vector conservation equation can be formulated, which relates the volume density $\boldsymbol{r} \times \boldsymbol{\omega}$ of moment of whirl to a suitable tensor flux. When moment of whirl is generated on the axis in axisymmetric flow it is only its axial component that is of relevance. Generation of this *axial component of moment of whirl* is identical with generation of the second (σ^2) moment of ring circulation. A conservation principle for the axial component of moment of whirl has been formulated previously by Pillow (1970). It can be derived directly, after some manipulation, from the covariant azimuthal component of (2.1). For swirl-free flows, devoid of potential-flow causes, it can be put in the form

$$\frac{\partial m}{\partial t} + \operatorname{div}\,[m(\boldsymbol{u} + 2\boldsymbol{q}_0) - \nu\, \boldsymbol{\nabla} m + \{2\boldsymbol{u}\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{u})\,\boldsymbol{I}\}\cdot\hat{\boldsymbol{x}}] = 0, \tag{4.7}$$

where m is the volume density of the axial component of moment of whirl. Here $m = \sigma^2 l$ and

$$\boldsymbol{u} = u\hat{\boldsymbol{x}} + v\hat{\boldsymbol{\sigma}}. \tag{4.8}$$

Equation (4.7) needs modification when, for instance, potential-flow causes alone are present, since it would call for a flux vector

$$(u^2-v^2)\,\hat{\boldsymbol{x}}+2uv\hat{\boldsymbol{\sigma}} \tag{4.9}$$

when no ring circulation has been generated. As pointed out by Paull (1982), this anomaly may be removed by noting that, in potential flow, the flux vector (4.9) is solenoidal and so allows the flux vector

$$(u_{\mathrm{p}}^2-v_{\mathrm{p}}^2)\,\hat{\boldsymbol{x}}+2u_{\mathrm{p}}\,v_{\mathrm{p}}\,\hat{\boldsymbol{\sigma}}, \tag{4.10}$$

generated by the velocity field

$$u_{\mathrm{p}}\,\hat{\boldsymbol{x}}+v_{\mathrm{p}}\,\hat{\boldsymbol{\sigma}}$$

of the volume-producing causes, to be subtracted from the flux vector in (4.7). Swirl-free conservation of the axial component of moment of whirl is then described by

$$\frac{\partial m}{\partial t}+\operatorname{div}\boldsymbol{N}=0, \tag{4.11}$$

where

$$\boldsymbol{N}=(\boldsymbol{u}+2\boldsymbol{q}_0)\,m-\nu\,\boldsymbol{\nabla}m+(u^2-v^2)\,\hat{\boldsymbol{x}}+2uv\hat{\boldsymbol{\sigma}}-[(u_{\mathrm{p}}^2-v_{\mathrm{p}}^2)\,\hat{\boldsymbol{x}}+2u_{\mathrm{p}}\,v_{\mathrm{p}}\,\hat{\boldsymbol{\sigma}}]. \tag{4.12}$$

It should be noted that all terms in $\boldsymbol{N}$ now concern ring-circulation-dependent quantities. The quadratic velocity terms, in particular, now only contain contributions amounting to interactions between the ring-circulation and potential-flow fields or the ring-circulation field and itself; the decomposition

$$\boldsymbol{u}=\boldsymbol{u}_{\mathrm{p}}+\boldsymbol{u}_l \tag{4.13}$$

of the velocity field into its potential flow and ring-circulation-induced velocities demonstrates

$$\begin{aligned}\boldsymbol{N}=\sigma^2l(\boldsymbol{u}_{\mathrm{p}}+\boldsymbol{u}_l)+2\sigma^2l\boldsymbol{q}_0-\nu\,\boldsymbol{\nabla}(\sigma^2l)+2(u_{\mathrm{p}}\,u_l-v_{\mathrm{p}}\,v_l)\,\hat{\boldsymbol{x}}&\\+2(v_{\mathrm{p}}\,u_l+u_{\mathrm{p}}\,v_l)\,\hat{\boldsymbol{\sigma}}+(u_l^2-v_l^2)\,\hat{\boldsymbol{x}}+2u_l\,v_l\,\hat{\boldsymbol{\sigma}}.&\end{aligned} \tag{4.14}$$

There are then no spurious potential-flow terms in the equation governing conservation of the second moment of ring circulation.

In conically similar viscous flow the flux vector $\boldsymbol{N}$ is given by

$$\begin{aligned}\boldsymbol{N}=\frac{\nu^2}{r^2}\hat{\boldsymbol{r}}\left[(3-f')\,(1-\mu^2)\,g-\frac{\mu}{1-\mu^2}\{f^2-f_{\mathrm{p}}^2\}-\{f^2-f_{\mathrm{p}}^2\}'+\mu\,\{(f')^2-(f_{\mathrm{p}}')^2\}\right]&\\+\frac{\nu^2}{r^2}\frac{\hat{\boldsymbol{\theta}}}{(1-\mu^2)^{\frac{1}{2}}}[(1-\mu^2)^2g'-(1-\mu^2)fg-\{f^2-f_{\mathrm{p}}^2\}+\mu\{f^2-f_{\mathrm{p}}^2\}'+(1-\mu^2)\,\{(f')^2-(f_{\mathrm{p}}')^2\}],&\end{aligned} \tag{4.15}$$

where νrf_{p} is the potential-flow stream function ($f_{\mathrm{p}}''=0$) arising from the volume-producing causes.

The inverse-square radial dependence of $\boldsymbol{N}$ indicates that not only is the axial component of moment of whirl conserved overall, but that the radial and transverse rates of discharge of $\boldsymbol{N}$ are separately conserved. There is no interchange between the two components. If $\boldsymbol{N}$ is rewritten as

$$\boldsymbol{N}=-\frac{\nu^2}{r^2}\chi'(\mu)\,\hat{\boldsymbol{r}}+\frac{\nu^2k\hat{\boldsymbol{\theta}}}{r^2(1-\mu^2)^{\frac{1}{2}}}, \tag{4.16}$$

the radial rate of discharge of the axial component of moment of whirl, in a thin conical annulus of μ-width $\mathrm{d}\mu$ neighbouring $\mu=\mu_0$, depends solely upon the rate

$-\nu^2\chi'(\mu_0)\,d\mu$ at which it is being discharged into that region by the singularity generating it at the origin. Likewise, the transverse rate of discharge of the axial component of moment of whirl, in the thin spherical shell between $r = r_0$ and $r = r_0 + dr$, takes place in meridional planes along lines of longitude, and depends solely upon the rate $2\pi\nu^2 k/r_0$ per unit radial thickness at which it is being discharged into the spherical shell from one half-axis. The constant k characterizes the strength of this transverse flow, which originates from an antisymmetric distribution of the axial component of moment of whirl sources on the axis of symmetry. A flux function X for the axial component of moment of whirl is consequently given by

$$X = \nu^2[\chi(\mu) - k\ln r], \tag{4.17}$$

where

$$\chi(\mu) = \int_{\mu}^{1} d\xi \left[(3-f')(1-\xi^2)g - \frac{\xi}{1-\xi^2}\{f^2 - f_p^2\} - \{f^2 - f_p^2\}' + \xi\{(f')^2 - (f_p')^2\}\right]. \tag{4.18}$$

In (4.17) a harmless artificial lengthscale has been introduced by taking $X = 0$ at $r = 1$, $\mu = 1$.

The strength of the point source of the axial component of moment of whirl is given by the first term in (4.17), as

$$\begin{aligned} L &= 2\pi\nu^2[\chi(-1) - \chi(1)] \\ &= 2\pi\nu^2 \int_{-1}^{1} d\xi \left[(3-f')(1-\xi^2)g - \frac{\xi}{1-\xi^2}\{f^2 - f_p^2\} - \{f^2 - f_p^2\}' + \xi\{(f')^2 - (f_p')^2\}\right]. \end{aligned} \tag{4.19}$$

The second term in (4.17) shows that the axial component of moment of whirl is emitted from the right half-axis of symmetry at the rate

$$2\pi\nu^2 k/r \tag{4.20}$$

per unit length. The strength of this cause is thus characterized by

$$K = 2\pi\nu^2 k, \tag{4.21}$$

which with the help of (4.1) and (4.15) can be rewritten as

$$K = 2\pi\nu^2[(C-A) + f_p(1)f_p(-1)], \tag{4.22}$$

where, in order to match the potential-flow volume-producing causes,

$$f_p(\pm 1) = f(\pm 1). \tag{4.23}$$

The relations (4.22), (4.4) and (4.5) serve to relate the integration constants A, B and C to the three physical constants M_{+1}, M_{-1} and K, which describe the strengths of the two uniform half-line sources of volume and the antisymmetric distribution of sources of axial component of moment of whirl. The problem of finding the solution $f(\mu)$ of (4.1) is then well-posed, since the further constant of integration involved is specified by the integral condition (4.19) in terms of the physical constant L, which describes the strength of the point source of axial component of moment of whirl at the origin 0.

A list summarizing this quantitative characterization of swirl-free conically similar viscous flows in terms of axial causes appears in table 1.

Conservation principle	Volume density, flux function	Non-dimensional governing equations	Axial cause(s), strength(s) of production, dimension of production	Expression(s) for cause strength(s) in terms of the non-dimensional conically similar viscous-flow solutions	Relations imposed as a result of specifying the strength of a cause
Volume	1, $\psi = \nu r f(\mu)$	$g = -f''(\mu)$	Uniform half-line source of volume on $\mu = \pm 1$, $M_{\pm 1}$, $\dfrac{\text{volume}}{\text{length} \times \text{time}}$	$M_{\pm 1} = \mp 2\pi\nu f(\pm 1)$	$A + C = [f(1) - f(-1)] - \frac{1}{4}[f^2(1) + f^2(-1)]$; $B = [f(1) + f(-1)] \times [1 - \frac{1}{4}\{f(1) - f(-1)\}]$
Ring circulation	$\dfrac{l}{2\pi} = \dfrac{\nu g(\mu)}{2\pi r^3}$, $\dfrac{-B}{2\pi} = \dfrac{-\nu^2\beta(\mu)}{2\pi r^2}$	$(1-\mu^2)f' + 2\mu f - \frac{1}{2}f^2 = A\mu^2 + B\mu + C$ (*)	Point source of axial component of moment of whirl, L; Antisymmetric distribution of axial component of moment of whirl sources with line density inversely proportional to distance from origin, K, $\dfrac{\text{length}^2 \times \text{circulation}}{\text{time}}$	$L = 2\pi\nu^2[\chi(-1) - \chi(1)]$, where $\chi = \displaystyle\int_\mu^1 \left[\begin{matrix} (3-f')(1-\mu^2)g - \dfrac{\mu}{1-\mu^2}\{f^2 + \tau^2 - f_p^2\} \\ -\{f^2 - f_p^2\}' \quad + \mu\{(f')^2 - (f_p')^2\} \end{matrix} \right] d\mu$ and $f_p = \frac{1}{2}[f(-1)(1-\mu) + f(1)(1+\mu)]$; $K = 2\pi\nu^2 k = 2\pi\nu^2[2(C-A) + f(1)f(-1)]$	Determines a particular integral of *; $A - C = -\frac{1}{2}k + \frac{1}{2}f(1)f(-1)$
Axial component of moment of whirl	$m = \sigma^2 l = \dfrac{\nu}{r}(1-\mu^2)g(\mu)$, $X = \nu^2[\chi(\mu) - k \ln r]$				

TABLE 1. Summary of conserved quantities, flux functions and axial causes for swirl-free conically-similar viscous flows

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